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# The Ernst equation and the Riemann–Hilbert problem on hyperelliptic Riemann surfaces

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**Abstract**

We discuss a class of solutions to the Ernst equation (the stationary axisymmetric Einstein equations) obtained as solutions of a generalized scalar Riemann–Hilbert problem on a hyperelliptic Riemann surface. The singular structure of these solutions is studied for arbitrary genus of the Riemann surface. A subclass is given for which the Ernst potential is everywhere regular besides at a contour that can be identified with the surface of a body of revolution. It turns out that the recently discussed rigidly rotating dust disk belongs to this class.

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The metric describing the exterior (i.e. the vacuum region) of an isolated axisymmetric, stationary rotating body can be written in the Weyl–Lewis–Papapetrou form (see [1])

$$ds^2 = -e^{2U}(dt + a\, d\phi)^2 + e^{-2U}(e^{2k}(d\rho^2 + d\zeta^2) + \rho^2 d\phi^2), \tag{1}$$

where $\rho$ and $\zeta$ are Weyl's canonical coordinates and $\partial_t$ and $\partial_\phi$ are the two commuting asymptotically time-like and space-like Killing vectors, respectively. In this case, the vacuum field equations are equivalent to the Ernst equation for the Ernst potential $f$ where $f = e^{2U} + ib$ and the real function $b$ is related to the metric functions via $b_z = -(i/\rho)e^{4U} a_z$.

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Here the complex variable $z$ stands for $z = \rho + \mathrm{i}\zeta$. With these settings, the Ernst equation reads

$$f_{z\bar{z}} + \frac{1}{2(z+\bar{z})}(f_{\bar{z}} + f_z) = \frac{2}{f+\bar{f}} f_z f_{\bar{z}} , \tag{2}$$

where a bar denotes complex conjugation in $\mathbb{C}$. With a solution of the Ernst equation, the metric function $U$ follows directly from the definition of the Ernst potential whereas $a$ and $k$ can be obtained from $f$ via quadratures.

The typical problem one has to treat in the context of the Ernst equation is to find the exterior solution for isolated matter distributions which is possible if an interior solution like the Wahlquist solution [2] is known, or if surface-like distributions are considered. This is equivalent to the solution of a boundary value problem where the Ernst potential has to be regular outside the contour that corresponds to the surface of the body. Recent work by Neugebauer and Meinel [3,4] suggests that such problems can be solved with the help of Riemann–Hilbert techniques. They succeeded in transforming the boundary value problem for the rigidly rotating dust disk (with radius $\rho_0$ and dust parameter $\mu_{\mathrm{d}}$) into a Riemann–Hilbert problem. Subsequently [5] they were able to reduce the resulting matrix problem to a scalar one on a hyperelliptic Riemann surface of genus 2.

The question we address here is to which extent Riemann–Hilbert techniques on hyperelliptic Riemann surfaces are useful in the construction of solutions to boundary value problems for the Ernst equation.

The remarkable feature of Eq. (2) is that it belongs to a class of completely integrable nonlinear equations. These are studied as the integrability condition of an overdetermined linear differential system which contains an additional variable, the so-called spectral parameter $K$. In the case under consideration, we use the linear system of [6] for a $2 \times 2$-matrix $\Phi$. Here the spectral parameter lives on a family of Riemann spheres $\mathcal{L}$ parametrized by $z$ and $\bar{z}$ and given by $\mu^2 = (K - \mathrm{i}\bar{z})(K + \mathrm{i}z)$.

The existence of the spectral parameter makes it possible to construct solutions to the Ernst equation by prescribing the singular structure of the matrix $\Phi$ with respect to $K$ (poles, essential singularities and cuts). The structure of the singularities has to be in accordance with a set of axioms that ensure that $\Phi$ is a solution of the linear system for some Ernst potential $f$. Furthermore, zeros of the determinant of $\Phi$ can be included in the formalism by introducing a two-sheeted covering $\hat{\mathcal{L}}$ of $\mathcal{L}$, as was done in [7–9] for the construction of finite gap solutions to the Ernst equation where the matrix $\Phi$ has poles and essential singularities.

The branch points $E_i$, $F_i$ of the covering are the zeros of the determinant if the components of $\Phi$ are regular there. They are subject to the reality condition $E_i, F_i \in \mathbb{R}$ or $E_i = \bar{F}_i$. The automorphism group $\sigma$ of $\hat{\mathcal{L}}$ turns out to be nontrivial. Therefore we may construct the hyperelliptic Riemann surface $\mathcal{L}_{\mathrm{H}} = \hat{\mathcal{L}}/\sigma$ given by $\mu^2 = (K - \mathrm{i}\bar{z})(K + \mathrm{i}z)\prod_{i=1}^{g}(K - E_i)(K - F_i)$, admitting an antiholomorphic involution, given by $\bar{P} = (\bar{K}, \mu(\bar{K}))$.

Thus it is possible to construct solutions to the Ernst equation on $\mathcal{L}_{\mathrm{H}}$ by prescribing the poles, essential singularities and cuts of $\Phi$, in other words we pose a generalized matrix Riemann–Hilbert problem, see [10]. Since the latter cannot be solved in general, we restrict

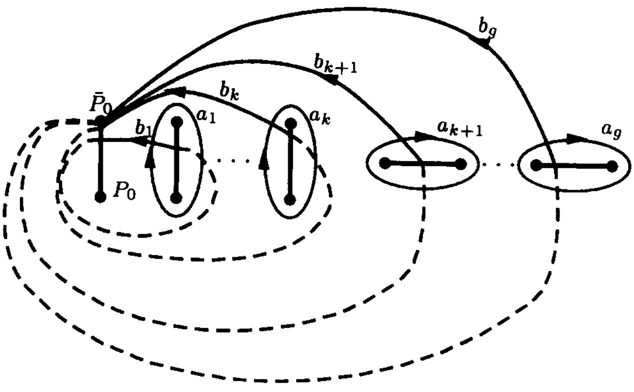


Fig. 1. The homology basis for $\mathcal{L}_{\mathrm{H}}$.

ourselves to the case of a scalar Riemann–Hilbert problem on $\mathcal{L}_{\mathrm{H}}$ that was solved by Zverovic [11] via quadratures. The key step of the solution is the construction on $\mathcal{L}_{\mathrm{H}}$ of an analog to the Cauchy kernel in the complex plane, e.g. $[(\mu(\tau)+\mu(K))/2\mu(\tau)(\tau-K)]\,\mathrm{d}\tau$ with $P=(K,\mu(K))\in\mathcal{L}_{\mathrm{H}}$ being fixed and representing the poles of the kernel, and $(\tau,\mu(\tau))\in\mathcal{L}_{\mathrm{H}}$. We introduce the standard quantities associated with a Riemann surface, namely with the cut system of Fig. 1, the $g$ normalized differentials of the first kind $\mathrm{d}\omega_i$ defined by $\oint_{a_i}\mathrm{d}\omega_j = 2\pi\mathrm{i}\delta_{ij}$ and the Abel map $\omega_i(P)=\int_{P_0}^{P}\mathrm{d}\omega_i$ with $P_0=(-\mathrm{i}z,0)$ (sometimes simply denoted by $-\mathrm{i}z$). Furthermore, we define the Riemann matrix $\Pi$ with the elements $\pi_{ij}=\oint_{b_i}\mathrm{d}\omega_j$ and the theta function $\Theta\left[{\alpha \atop \beta}\right](z)=\sum_{N\in\mathbb{Z}^g}\exp\{\frac{1}{2}\langle\Pi(N+\alpha),(N+\alpha)\rangle+\langle(z+2\pi\mathrm{i}\beta),(N+\alpha)\rangle\}$ with integer characteristic $\left[{\alpha \atop \beta}\right]\in\frac{1}{2}\mathbb{Z}^g/\mathbb{Z}^g$ ($\langle N,z\rangle=\sum_{i=1}^{g}N_i z_i$). The normalized (all $a$-periods zero) Cauchy analog with poles in $P$ and $P_0$ will be denoted by $\mathrm{d}\omega_{PP_0}$.

We have the following theorem.

**Theorem 1.** *Let $\Gamma$ be a closed, piecewise smooth contour on $\mathcal{L}_{\mathrm{H}}$ such that with $P\in\Gamma$ also $\overline{P}\in\Gamma$, and $G$ a finite nonzero function on $\Gamma$ subject to $G(\overline{P})=\overline{G}(P)$. Let $\chi(P)$ obey the jump condition $\chi^{+}=G\chi^{-}$ on $\Gamma$ with $\chi(\infty^{-})=1$. Let $\Omega(P)$ be an arbitrary linear combination of normalized Abelian integrals of the second kind and of integrals of the third kind with poles at the real branch points $E_i$ of order $\frac{1}{2}$ and poles at real $F_i$ of order $-\frac{1}{2}$. The b-periods of $\Omega$ are $b_i$ where $\overline{\Omega}(P)=\Omega(\overline{P})$ and the singularities of the integrals of the second kind do not coincide with the $E_i$ or $F_i$. Let $P_0\in\mathcal{L}_{\mathrm{H}}$ ($\rho\neq 0$) be fixed and not coinciding with the singularities of $\chi$ or the branch points. Choose a characteristic $\left[{\alpha \atop \beta}\right]$ such that $\Theta\left[{\alpha \atop \beta}\right](\omega(P))=0$ at the points $P=E_i$ or $P=F_i$ only, and has the same properties under complex conjugation as the theta function with the characteristic $\left[{0 \atop 0}\right]$. Then*

$$\chi(P)=\chi_0\,\frac{\Theta\left[{\alpha \atop \beta}\right](\omega(P)+u+b)}{\Theta\left[{\alpha \atop \beta}\right](\omega(P))}\exp\left\{\Omega(P)+\frac{1}{2\pi\mathrm{i}}\int_{\Gamma}\ln G(\tau)\,\mathrm{d}\omega_{PP_0}(\tau)\right\} \quad (3)$$

*is the solution to the scalar Riemann–Hilbert problem on $\mathcal{L}_H$ with a jump at $\Gamma$, essential singularities at the poles of the integrals of the second kind, and poles at the points $E_i$ or $F_i$ and the endpoints of $\Gamma$. Here the $u_i$ are the b-periods of the Cauchy integral, $\chi_0$ is a normalization constant, and the path of integration between $P_0$ and $P$ has to be the same for all integrals.*

*Proof (sketch).* It can be easily seen that $\Phi(P)$ is a single-valued function on $\mathcal{L}_H$: the difference of two paths of integration between $P_0$ and $P$ can be represented as a linear combination of the $a$-and $b$-cuts since they are a basis of the homology. Therefore, by a change of the integration path, the theta quotient will be multiplied by $\exp(-\langle N, b+u\rangle)$ but this term is just compensated by the integrals in the exponent. The analytic properties of $\chi$ follow from the definition of the Abelian differentials and the properties of the theta function. □

The underlying axiomatics ensures that the function $f = \chi(\infty^+)$ given by

$$f = \frac{\Theta\left[{\alpha \atop \beta}\right](\omega(\infty^+) + u + b)}{\Theta\left[{\alpha \atop \beta}\right](\omega(\infty^-) + u + b)} \times \exp\left\{\Omega(\infty^+) - \Omega(\infty^-) + \frac{1}{2\pi \mathrm{i}} \int_\Gamma \ln G(\tau)\, \mathrm{d}\omega_{\infty^+\infty^-}(\tau)\right\} \tag{4}$$

is a regular solution to the Ernst equation if $P_0$ is subject to the conditions of Theorem 1 and $\Theta\left[{\alpha \atop \beta}\right](\omega(\infty) + u + b) \neq 0$.

This class contains for constant $G$ the finite gap solutions of [7–9]. There, the so-called Baker–Akhiezer function (a function with essential singularities and poles) was constructed for the Ernst equation. This function plays the central role for the nonlinear evolution equations for which initial value problems are posed (see [12]).

However, we recall that in the case of the Ernst equation, one is confronted with boundary value problems. As already mentioned above, Neugebauer and Meinel were able to transform the boundary value problem for the rigidly rotating disk of dust to a matrix Riemann–Hilbert problem on the Riemann sphere $S^2$ and subsequently to a scalar Riemann–Hilbert problem on a hyperelliptic surface of genus 2 with branch points $E_1 = \sqrt{(\mathrm{i} - \mu_\mathrm{d})/\mu_\mathrm{d}}$, $E_2 = -F_1$ and $F_i = \bar{E}_i$. This solution just falls in the class above, where the contour $\Gamma$ is the covering of the imaginary axis between $-\mathrm{i}\rho_0, \mathrm{i}\rho_0$ in the +–sheet, the characteristic reads

$$\frac{1}{2}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$

and $G$ is given by

$$G = \left(\sqrt{1 + \mu_\mathrm{d}^2(1+X^2)^2} + \mu_\mathrm{d}(1+X^2)\right)^2$$

(dimensionless coordinates $\zeta/\zeta_0$, $\xi/\xi_0$ are used).

It was observed in [13] that this solution is part of a large class which can be obtained by prescribing one of the periods of the integrals in the exponent of (4) on $\mathcal{L}_H$ without real branch points as an arbitrary solution of the (axisymmetric) Laplace equation. Then the other periods as well as these integrals can be calculated from the so-called Picard–Fuchs equations together with some boundary conditions. The integrability condition of this Picard–Fuchs system is just the Laplace equation which implies that all periods are necessarily solutions of the Laplace equation. Later this class was extended in [14] to the case of real branch points. Since any solution to the Laplace equation can be written as periods of an (possibly infinite) linear combination of Abelian integrals of the second kind and a Cauchy integral over some contour $\Gamma$, the class given in [14] is equivalent to (4) in this sense. Therefore the results of [13,14] can be related to a well-posed mathematical problem, a Riemann–Hilbert problem on a hyperelliptic Riemann surface.

The main aim of the present paper is however to clarify if exterior solutions for isolated relativistic bodies can be identified within this class. Since the vacuum solutions considered in this context will be singular at the boundary of the matter region, one has to identify the singularities of the Ernst potential (4) with this boundary which will be a contour in the $(\rho, \zeta)$-plane. The advantage of the approach used here is that the possible singularities of $f$ can be easily identified. These are exactly the points where the conditions of Theorem 1 do not hold: the axis $\rho = 0$ (this is a reminescent of the fact that the axisymmetric Laplace operator entering the Ernst equation is singular there), the branch points $E_i$ and $F_i$, the singularities of the Abelian integrals in the exponent of (3), and a diverging normalization constant $\chi_0$. The latter simply leads to a pole of the Ernst potential. For the remaining points, we have the following theorem.

**Theorem 2.** *Let $f$ be the Ernst potential represented in the form* (4). *Then we have:*

(1) *$f$ has essential singularities at the points where $P_0$ coincides with the singularities of the Abelian integrals of the second kind in $\Omega$.*

(2) *The Ernst potential may have singularities where $P_0$ coincides with the end points of the contour $\Gamma$. $f$ is bounded if $P_0$ approaches the contour $\Gamma$ at the remaining points. Its limiting value will depend on the side from which the contour is approached.*

(3) *Let $\Sigma'$ be the Riemann surface given by $\mu'^2 = \prod_{i=1}^{g}(\tau - E_i)(\tau - F_i)$. A prime denotes that the primed quantity is taken on this surface. Let*

$$\Theta'\begin{bmatrix}\alpha'\\ \beta'\end{bmatrix}(\omega'|_{\zeta^+}^{\infty^+} + u' + b') + (-1)^{\varepsilon}\exp(-(\omega'_g(\infty^+) + u_g + b_g))\Theta'\begin{bmatrix}\alpha'\\ \beta'\end{bmatrix}(\omega'|_{\zeta^-}^{\infty^\pm} + u' + b') \neq 0 \tag{5}$$

*on the axis where the theta characteristic $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ is written as*

$$\begin{bmatrix}\alpha' & \frac{1}{2}\\ \beta' & \frac{1}{2}\epsilon\end{bmatrix} \quad \textit{with } \epsilon = 0, 1.$$

*Then the Ernst potential is regular on the axis besides at the singularities of $\Omega$, points of $\Gamma$, and branch points $E_i$, $F_i$.*

(4) *For $\Theta'\left[\begin{smallmatrix}\alpha'\\ \beta'\end{smallmatrix}\right](u'+b')\neq 0$, the asymptotic behavior of $f$ for large $\rho^2+\zeta^2$ is given by $f=1-m/\sqrt{\rho^2+\zeta^2}$ where $m$ is a complex constant.*

(5) *$f$ is bounded but in general not differentiable if $P_0$ coincides with real branch points $E_i$ or $F_i$.*

(6) *Let $\Sigma''$ be the Riemann surface given by $\mu''^2=\prod_{i=1}^{g-1}(\tau-E_i)(\tau-F_i)$. A double prime denotes that the corresponding quantity is taken on this surface. Let $E_i, F_i\notin\Gamma$ be nonreal branch points. For $g=1$, $f$ is regular at points where $P_0$ coincides with $E_1$ or $F_1$. If $g\geq 2$ then $f$ is regular at points where $P_0$ coincides with $E_i$ or $F_i$ if and only if the following holds:*

$$\begin{aligned}0\neq\Theta''&\begin{bmatrix}\alpha''\\ \beta''\end{bmatrix}(y''+\epsilon_{11}\pi_{g-1}+\epsilon_{12}\pi_g)\\&+2\epsilon_{11}\exp\{-(y_{g-1}+\epsilon_{12}\pi_{(g-1)g}+2\pi\mathrm{i}\epsilon_{21})\}\\&\times\Theta''\begin{bmatrix}\alpha''\\ \beta''\end{bmatrix}(y''-\epsilon_{11}\pi_{g-1}+\epsilon_{12}\pi_g)\\&+2\epsilon_{12}\exp\{-(y_g+\epsilon_{11}\pi_{(g-1)g}+2\pi\mathrm{i}\epsilon_{22})\}\\&\times\Theta''\begin{bmatrix}\alpha''\\ \beta''\end{bmatrix}(y''+\epsilon_{11}\pi_{g-1}-\epsilon_{12}\pi_g)\\&+4\epsilon_{11}\epsilon_{12}\exp\{-(y_{g-1}+y_g+2\pi\mathrm{i}(\epsilon_{21}+\epsilon_{22}))\}\\&\times\Theta''\begin{bmatrix}\alpha''\\ \beta''\end{bmatrix}(y''-\epsilon_{11}\pi_{g-1}-\epsilon_{12}\pi_g),\end{aligned}\tag{6}$$

*where $y=u+b\pm\omega(\infty^+)$ and the theta characteristic $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ is written as*

$$\begin{bmatrix}\alpha'' & \epsilon_{11} & \epsilon_{12}\\ \beta'' & \epsilon_{21} & \epsilon_{22}\end{bmatrix}.$$

*Proof (sketch).*

(1) Follows from the definition of the Abelian integrals.

(2) Follows from the fact that $P_0$ is a branch point of $\mathcal{L}_H$. Consequently, the sign of the root one has to choose in the evaluation of the contour integrals depends on how $P_0$ approaches the contour.

(3) The Ernst potential is best evaluated on the axis in a cut system where the cut $a_g$ encircles $[P_0, \overline{P}_0]$ and where all $b$-cuts start at the cut $[E_1, F_1]$. Then it is a standard result of Fay [15] that in the limit $\rho\to 0$ the differential of the first kind $\mathrm{d}\omega_g$ becomes the differential of the third kind $\mathrm{d}\omega'_{\zeta^-\zeta^+}$ on $\Sigma'$ plus an expansion in powers of $\rho$ that contains Abelian integrals of the second kind on $\Sigma'$. The theta function on $\mathcal{L}_H$ thus breaks down to the sum of two theta functions on $\Sigma'$ that does not vanish if (5) holds. This proves the statement.

(4) The asymptotic behavior of $f$ can be directly calculated from the axis potential.

(5) Using the cut system of (3) and putting $P_0 = E_g + x$ with $x = \delta e^{i\phi}$ where $\delta \in \mathbb{R}^+$, $\phi \in \mathbb{R}$, one recognizes that the expansion of $f$ in the vicinity of the real branch points contains powers of $\sqrt{x}$ and $\sqrt{\bar{x}}$ with in general $\phi$-dependent coefficients. Only the limiting value $P_0 \to E_g$ exists.

(6) Without loss of generality we may restrict ourselves to $E_g$. For the other branch points, the considerations are similar. The investigation resembles the one on the axis: choose a cut system where $a_g$ encircles $[\overline{P}_0, F_g]$ and $a_{g-1}$ the cut $[P_0, E_g]$. Then $d\omega_g$, respectively, $d\omega_{g-1}$ become differentials of the third kind on $\Sigma''$ with poles at $F_g^+, F_g^-$, respectively, at $E_g^+, E_g^-$. The evaluation of $f$ is as on the axis and the limiting value exists if (6) holds. If $E_g \in \Gamma$, then $u_g$ or its derivatives will diverge for $P_0 = E_g$. □

The results from above can be given an obvious physical interpretation: the essential singularities of $f$ correspond to line singularities of the metric function $U$, the real branch points to topological defects on the axis. It is doubtful that they have any physical application. In contrast, the contour $\Gamma_z$ given by $P_0 \in \Gamma$ can be related to the surface of an isolated body since the Ernst potential is only discontinuous there. The requirement $\Theta'\left[\begin{smallmatrix}\alpha'\\ \beta'\end{smallmatrix}\right](u' + b') \neq 0$ is independent of the physical coordinates and can only be violated if a uniquely determined relation between the contour integrals and the Riemann surface holds. If the branch points $E_i$ and $F_i$ and the jump function $G(\tau)$ depend upon a single parameter, there will be a whole range of this parameter for which this condition holds. A similar discussion is possible for the conditions appearing in (3) and (6).

From Theorem 2, one distinguishes a physically interesting class of solutions as follows.

**Corollary 3.** *Let the branch points $E_i$, $F_i$ $(i = 1, \ldots, g)$ of $\mathcal{L}_H$ be nonreal, $\chi_0$ be finite, and $\Gamma$ be a smooth, closed contour on $\mathcal{L}_H$ such that* (5) *and* (6) *are fulfilled. Then $f$ is a regular solution of the Ernst equation everywhere outside $\Gamma_z$.*

This suggests together with the fact that the solution for the rigidly rotating dust disk belongs to this class that further physically interesting solutions can be identified there, and stresses the importance of Riemann–Hilbert techniques in the context of solutions to boundary value problems for the Ernst equation.

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## References

[1] D. Kramer, H. Stephani, E. Herlt and M. MacCallum, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, 1980).
[2] H. D. Wahlquist, Interior solution for a finite rotating body of perfect fluid, Phys. Rev. 172 (1968) 1291.
[3] G. Neugebauer and R. Meinel, The Einsteinian gravitational field of the rigidly rotating disk of dust, Astrophys. J. 414 (1993) L97.
[4] G. Neugebauer and R. Meinel, General relativistic gravitational field of the rigidly rotating disk of dust: Axis potential, disk metric, and surface mass density, Phys. Rev. Lett. 73 (1994) 2166.
[5] G. Neugebauer and R. Meinel, General relativistic gravitational field of a rigidly rotating disk of dust: Solution in terms of ultraelliptic functions, Phys. Rev. Lett. 75 (1995) 3046.

[6] G. Neugebauer, Bäcklund transformations of axially symmetric stationary gravitational fields, J. Phys. A 12 (1979) L67.
[7] D.A. Korotkin, Finite-gap solutions of the stationary axisymmetric Einstein equations in vacuum, Theoret. and Math. Phys. 77 (1989) 1018.
[8] D.A. Korotkin, Algebraic geometric solutions of Einstein's equations: Some physical properties, Comm. Math. Phys. 137 (1991) 383.
[9] D.A. Korotkin, Elliptic solutions of stationary axisymmetric Einstein equation, Classical Quantum Gravity 10 (1993) 2587.
[10] O. Richter and C. Klein, Algebro-geometric approach to the Ernst equation I: Mathematical preliminaries, AGG 46 (to appear in Banach Center Publications).
[11] S.I. Zverovich, Boundary value problems in the theory of analytic functions in Hölder classes on Riemann surfaces, Russian. Math. Surveys 26 (1971) 117–192.
[12] E.D. Belokolos, A.I. Bobenko, V.Z. Enolskii, A.R. Its and V.B. Matveev, *Algebro-Geometric Approach to Nonlinear Integrable Equations* (Springer, Berlin, 1994).
[13] R. Meinel and G. Neugebauer, Solutions to Einstein's field equations related to Jacobi's inversion problem, Phys. Lett. A 210 (1996) 160.
[14] D.A. Korotkin, Some remarks on finite gap solutions of the Ernst equation, preprint gr-qc/9602021.
[15] J.D. Fay, *Theta Functions on Riemann Surfaces*, Lecture Notes in Mathematics, Vol. 352 (Springer, New York, 1973).